Chapter 5

Legendre Polynomials

$P_m(x)$ is a Legendre Polynomials which is a solution of Legendre equation

$(1-x^2)y''-2xy'+m(m+1)y=0$ in the form

$$P_m(x)=\sum_{r=0}^{[m/2]}(-1)^r\frac{(2m-2r)!}{2^m\,r!(m-r)!(m-2r)!}x^{m-2r}$$

we find that

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x)=\frac{1}{2}(3x^2-1)$$

$$P_3(x)=\frac{1}{2}(5x^2-3x)$$

$$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$$

Generating Function for the Legendre polynomials:

$$\frac{1}{\sqrt{(1-2tx+t^2)}}=\sum_{m=0}^{\infty}t^mP_m(x)\quad if\ |t|<1\ and\ |x|\le 1$$

Further Expressions for the Legendre polynomials:

(Rodrigues's Formula)

$$P_m(x)=\frac{1}{2^m\,m!}\cdot\frac{d^m}{dx^m}(x^2-1)^m$$

from this formula we find that

$P_0(x)=1$ and $P_1(x)=\frac{1}{2}\frac{d}{dx}(x^2-1)=x$

$P_2(x)=\frac{1}{2^2 2!}\cdot\frac{d^2}{dx^2}(x^2-1)^2=\frac{1}{2}(3x^2-1)$ and so on

(Laplace's Integral Representation) $P_m(x)=\frac{1}{\pi}\int_0^{\pi}\left(x+\sqrt{x^2-1}\,\cos\theta\right)^m d\theta$

Explicit Expressions for and Special Values of The Legendre Polynomials:

(i) $P_m(1) = 1$

(ii) $P_m(-1) = (-1)^m$

(iii) $\frac{d}{dx} P_m(1) = \frac{1}{2} m(m+1)$

(iv) $\frac{d}{dx} P_m(-1) = (-1)^{m-1} \frac{1}{2} m(m+1)$

(v) $P_{2m}(0) = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2}$

(vi) $P_{2m+1}(0) = 0$

Orthogonal Properties of the Legendre Polynomials:

$$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2}{2m+1} & \text{if } m = n \end{cases}$$

Some Relation between the Legendre Polynomials and Their derivatives:

(Recurrence Relations):

Theorem (7):

(i) $$P'_m(x) = \sum_{r=0}^{\frac{1}{2}[m-1]} (2m - 4r - 1) P_{m-2r-1}(x)$$

(ii) $$xP_m(x) = \frac{m+1}{2m+1} P_{m+1}(x) + \frac{m}{2m+1} P_{m-1}(x)$$

(iii) $(m+1)P_{m+1}(x) - (2m+1)xP_m(x) + mP_{m-1}(x) = 0$

(iv) $P'_{m+1}(x) - P'_{m-1}(x) = (2m+1)P_m(x)$

(v) $xP'_m(x) - P'_{m-1}(x) = mP_m(x)$

(vi) $P'_m(x) - xP'_{m-1}(x) = mP_{m-1}(x)$

(vii) $(x^2 - 1)P'_m(x) = mxP_m(x) - mP_{m-1}(x)$

(viii) $(x^2 - 1)P'_m(x) = (m+1)P_{m+1}(x) - (m+1)xP_m(x)$

Solved Examples:

Example (1):

Show that $\int_{-1}^{1} x^2 P_{m+1}(x) P_{m-1}(x) dx = \frac{2m(m+1)}{(4m^2-1)(2m+3)}$

Deduce the value of $\int_{0}^{1} x^2 P_{m+1}(x) P_{m-1}(x)\, dx$

Solution:

We use theorem (8)ii to dispose of the x^2 appearing in the integrand, and we may then use the orthonormality property of theorem(5)

Thus

$$\int_{-1}^{1} x^2 P_{m+1}(x) P_{m-1}(x) dx$$

$$= \int_{-1}^{1} \left(xP_{m+1}(x)\right)\left(xP_{m-1}(x)\right) dx$$

$$= \int_{-1}^{1} \left\{\frac{m+2}{2m+3} P_{m+2}(x) + \frac{m+1}{2m+3} P_m(x)\right\}\left\{\frac{m}{2m-1} P_m(x) + \frac{m-1}{2m-1} P_{m-2}(x)\right\} dx$$

$$= \int_{-1}^{1} \frac{m+1}{2m+3} P_m(x) \frac{m}{2m-1} P_m(x) dx$$

{the other terms vanishing by theorem(5), since they are of the

form $\int_{-1}^{1} P_l(x) P_m(x) dx,\ l \neq m$ }

$$\int_{-1}^{1} x^2 P_{m+1}(x) P_{m-1}(x) dx = \frac{m(m+1)}{(2m+3)(2m-1)} \int_{-1}^{1} \left\{P_m(x)\right\}^2 dx$$

$$= \frac{m(m+1)}{(2m+3)(2m-1)} \cdot \frac{2}{2m+1} \{\text{by theorem(5)}\} = \frac{2m(m+1)}{(2m+3)(4m^2-1)}$$

we know that $P_m(x)$ is a polynomial of degree m , so that the above integrand is a polynomial of degree $2 + (m+1) + (m-1) = 2(m+1)$ i.e. of even degree

Thus the integrand is even, so that

$$\int_{-1}^{1} x^2 P_{m+1}(x) P_{m-1}(x) dx = 2\int_{0}^{1} x^2 P_{m+1}(x) P_{m-1}(x) dx$$

Hence

$$2\int_0^1 x^2 P_{m+1}(x)P_{m-1}(x)dx = \frac{2m(m+1)}{(2m+3)(4m^2-1)}$$

$$\therefore \int_0^1 x^2 P_{l+1}(x)P_{l-1}(x)dx = \frac{m(m+1)}{(2m+3)(4m^2-1)}$$

Example (2):

Evaluate $\int_{-1}^{1} P_m(x)dx$ when m is odd

Solution:

By theorem (8) we have $P'_{m+1}(x) - P'_{m-1}(x) = (2m+1)P_m(x)$ then

$$P_m(x) = \frac{1}{(2m+1)}\left\{P'_{m+1}(x) - P'_{m-1}(x)\right\}$$

$$\int_0^1 P_m(x)dx = \frac{1}{(2m+1)}\int_0^1 \left\{P'_{m+1}(x) - P'_{m-1}(x)\right\}dx$$

$$= \frac{1}{(2m+1)}\left[\left\{P_{m+1}(x) - P_{m-1}(x)\right\}\right]_0^1$$

$$= \frac{1}{(2m+1)}\left[\left\{P_{m+1}(1) - P_{m-1}(1)\right\} - \left\{P_{m+1}(0) - P_{m-1}(0)\right\}\right]$$

$$= \frac{1}{(2m+1)}\left[1 - 1 - \frac{(-1)^{(m+1)/2}}{2^{(m+1)}}\frac{(m+1)!}{\left[\left\{(m+1)/2\right\}!\right]^2} + \frac{(-1)^{(m-1)/2}}{2^{(m-1)}}\frac{(m-1)!}{\left[\left\{(m-1)/2\right\}!\right]^2}\right]$$

{by theorem (4)I and v, note that $m+1$and $m-1$are both even}

$$= \frac{1}{(2m+1)}\left[-\frac{(-1)(-1)^{(m-1)/2}}{.2^2 2^{(m-1)}}\frac{(m+1)m(m-1)!}{\left[(m+1)/2\left\{(m-1)/2\right\}!\right]^2} + \frac{(-1)^{(m-1)/2}}{2^{(m-1)}}\frac{(m-1)!}{\left[\left\{(m-1)/2\right\}!\right]^2}\right]$$

$$= \frac{1}{(2m+1)}\frac{(-1)^{(m-1)/2}}{2^{(m-1)}}\frac{(m-1)!}{\left[\left\{(m-1)/2\right\}!\right]^2}\left[1 - \frac{(-1)(m+1)m}{.2^2\left[\left\{(m+1)/2\right\}\right]^2}\right]$$

$$= \frac{1}{(2m+1)}\frac{(-1)^{(m-1)/2}}{2^{(m-1)}}\frac{(m-1)!}{\left[\left\{(m-1)/2\right\}!\right]^2}\left[1 + \frac{m}{m+1}\right]$$

$$= \frac{1}{(2m+1)}\frac{(-1)^{(m-1)/2}}{2^{(m-1)}}\frac{(m-1)!}{\left[\left\{(m-1)/2\right\}!\right]^2}\left[\frac{2m+1}{m+1}\right]$$

$$=\frac{(-1)^{(m-1)/2}}{2^{(m-1)}}\frac{(m-1)!}{[m+1]\left(\frac{m}{2}-\frac{1}{2}\right)!\left(\frac{m}{2}-\frac{1}{2}\right)!}$$

$$=\frac{(-1)^{(m-1)/2}(m-1)!}{2^m\left(\frac{m}{2}+\frac{1}{2}\right)\left(\frac{m}{2}-\frac{1}{2}\right)!\left(\frac{m}{2}-\frac{1}{2}\right)!}=\frac{(-1)^{(m-1)/2}(m-1)!}{2^m\left(\frac{m+1}{2}\right)!\left(\frac{m-1}{2}\right)!}$$

Example (3)

Expand the function $f(x)=\begin{cases}1 & 0<x<1\\ -1 & -1<x<0\end{cases}$ in Legendre series.

Solution

$$C_n=(n+\tfrac{1}{2})\int_{-1}^{1} f(x)P_n(x)dx$$

$$=\tfrac{1}{2}(2n+1)\int_{-1}^{1} f(x)P_n(x)dx$$

Since the function is an odd then Legendre series contains only odd indexed polynomial we have

$$C_{2n+1}=(4n+3)\int_0^1 f(x)P_{2n+1}(x)dx=(4n+3)\int_0^1 P_{2n+1}(x)dx$$

Use the fact that

$(2k+1)P_k=P'_{k+1}(x)-P'_{k-1}(x)$ with $\mathrm{k}=2\mathrm{n}+1$

$(4\mathrm{n}+3)P_{2n+1}=P'_{2n+2}(x)-P'_{2n}(x)$

With $P_{2n}(1)=1$ and $P_{2n}(0)=\dfrac{(-1)^n(2n)!}{2^{2n}(n!)^2}$

Then

$$C_{2n+1}=(4n+3)\int_0^1 P_{2n+1}dx=\int_0^1 \left[P'_{2n+2}(x)-P'_{2n}(x)\right]dx=\left[P_{2n+2}(x)-P_{2n}(x)\right]_0^1$$

$$=P_{2n}(0)-P_{2n+2}(0)+=\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}-\frac{(-1)^{n+1}(2n+2)!}{2^{2n+2}(n+1)!^2}$$

$$=\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}+\frac{(-1)^n(2n+2).(2n+1).(2n)!}{2^{2n}.2^2(n+1)^2.(n!)^2}$$

$$=\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}+\frac{(-1)^n.(2n+1).(2n)!}{2^{2n}.2(n+1).(n!)^2}$$

$$=\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\left(1+\frac{2n+1}{2(n+1)}\right)$$

$$=\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\left(\frac{2n+2}{2(n+1)}+\frac{2n+1}{2(n+1)}\right)=\left(\frac{(-1)^n(4n+3)}{2^{n+1}(n+1)}\right)\frac{(2n)!}{(n!)^2}$$

We have the expression

$$f(x)=\sum_{n=0}^{\infty}\left(\frac{(-1)^n(4n+3)}{2^{n+1}(n+1)}\right)\frac{(2n)!}{(n!)^2}P_{2n+1}(x)\quad 0<x<1$$

Example (4)

Expand the function $f(x)=|x|,\quad -1<x<1$ in Legendre series.

Solution

Now, $|x|$ is an even function and $P_r(x)$ is even if *r* is even and odd if *r* is odd. Hence if *r* is odd $|x|P_r(x)$ is odd, so that $\int_{-1}^{1}|x|P_r(x)dx=0$ hence $C_r=0$ On the other hand, if *r* is even $|x|P_r(x)$ is even and hence $\int_{-1}^{1}|x|P_r(x)dx=2\int_{0}^{1}xP_r(x)dx$

So that

$$C_r=(2r+1)\int_0^1 xP_r(x)dx=(2r+1)\int_0^1\left\{\frac{r+1}{2r+1}P_{r+1}(x)+\frac{r}{2r+1}P_{r-1}(x)\right\}dx$$

$$=\int_0^1\{(r+1)P_{r+1}(x)+r\,P_{r-1}(x)\}dx$$

Now, *r* is even, so that both *r* + 1 and *r*- 1 are odd, and we may use the result of example (2) to obtain

$$C_r = (r+1)\frac{(-1)^{r/2} r!}{2^{r+1}(\frac{1}{2}r+1)!(\frac{1}{2}r)!} + r\,\frac{(-1)^{r/2-1}(r-2)!}{2^{r-1}(\frac{1}{2}r)!(\frac{1}{2}r-1)!}$$

$$= \frac{(-1)^{r/2}(r-2)!}{2^{r-1}(\frac{1}{2}r)!(\frac{1}{2}r-1)!}\left\{\frac{(r+1)r(r-1)}{2^2(\frac{1}{2}r+1)(\frac{1}{2}r)} - r\right\}$$

$$= \frac{(-1)^{r/2}(r-2)!}{2^{r-1}(\frac{1}{2}r)!(\frac{1}{2}r-1)!}\left\{\frac{(r+1)(r-1)}{r+2} - r\right\}$$

$$= \frac{(-1)^{r/2+1}(r-2)!}{2^{r-1}(\frac{1}{2}r)!(\frac{1}{2}r-1)!}\frac{(2r+1)}{(r+2)}$$

Thus we have

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(4n+1)(2n-2)!}{2^{2n}(n+1)!(n-1)!} P_{2n}(x)$$

Example (5)

Use Generating Function to prove $(m+1)P_{m+1}(x) - (2m+1)xP_m(x) + mP_{m-1}(x) = 0$

Proof

Consider the generating function

$$\frac{1}{\sqrt{(1-2tx+t^2)}} = \sum_{n=0}^{\infty} t^n P_n(x) \qquad (1)$$

We differentiate with respect to *t*:

$$\frac{(x-t)}{\left(\sqrt{(1-2tx+t^2)}\right)^3} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1}$$

We multiply by $1 - 2xt + t^2$ and use (1)

$$\frac{(1-2tx+t^2)(x-t)}{\left(\sqrt{1-2tx+t^2}\right)^3} = \sum_{n=1}^{\infty} (1-2tx+t^2)nP_n(x)t^{n-1}$$

$$\frac{(x-t)}{\sqrt{1-2tx+t^2}} = \sum_{n=1}^{\infty} (1-2tx+t^2)nP_n(x)t^{n-1}$$

$$\sum_{n=0}^{\infty} (x-t)P_n(x)t^n = \sum_{n=1}^{\infty} (1-2tx+t^2)nP_n(x)t^{n-1}$$

$$\sum_{n=0}^{\infty} xP_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1}$$

$$= \sum_{n=1}^{\infty} nP_n(x)t^{n-1} - \sum_{n=1}^{\infty} 2nxP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1}$$

and after grouping the series

$$xP_0(x) - P_1(x) + \sum_{n=1}^{\infty} \left[(2n+1)xP_n(x) - (n+1)P_{n+1}(x) - nP_{n\text{-}1}(x)\right]t^n = 0$$

Equating the coefficient then

$$(2n+1)xP_n(x) - (n+1)P_{n+1}(x) - nP_{n-1}(x)$$

Exercise

(1) Show that $\boxed{\int_{-1}^{1} xP_m(x)P_{m-1}(x)dx = \frac{2m}{4m^2-1}}$

Answer:

Since $$xP_m(x) = \frac{m+1}{2m+1}P_{m+1}(x) + \frac{m}{2m+1}P_{m-1}(x)$$

Then

$$xP_m(x)P_{m-1}(x) = \frac{m+1}{2m+1}P_{m+1}(x)P_{m-1}(x) + \frac{m}{2m+1}P_{m-1}(x)P_{m-1}(x)$$

$$\int_{-1}^{1} xP_m(x)P_{m-1}(x)dx$$

$$= \int_{-1}^{1} \frac{m+1}{2m+1}P_{m+1}(x)P_{m-1}(x)dx + \int_{-1}^{1} \frac{m}{2m+1}P_{m-1}(x)P_{m-1}(x)dx$$

$$= \quad 0 \quad + \int_{-1}^{1} \frac{l}{2m+1} P_{m-1}(x) P_{m-1}(x) dx$$

$$= \frac{m}{2m+1} \cdot \frac{2}{2(m-1)+1} = \frac{2m}{(2m+1)(2m-1)} = \frac{2m}{4m^2-1}$$

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(2) Show that $\boxed{\int_{-1}^{1} (1-x^2) P_m'(x) P_n'(x) dx = \frac{2m(m+1)}{2m+1} \delta_{mn}}$

Answer:

Since $(x^2-1) P_m'(x) = mx P_m(x) - m P_{m-1}(x)$

Then $\quad (1-x^2) P_m'(x) = -mx P_m(x) + m P_{m-1}(x) \qquad (*)$

and

$$x P_m(x) = \frac{m+1}{2m+1} P_{m+1}(x) + \frac{m}{2m+1} P_{m-1}(x) \qquad (**)$$

from (**) in (*) we have

$$(1-x^2) P_m'(x) = -m\left(\frac{m+1}{2m+1} P_{m+1}(x) + \frac{m}{2m+1} P_{m-1}(x)\right) + m P_{m-1}(x)$$

$$= \frac{-m(m+1)}{2m+1} P_{m+1}(x) - \frac{m^2}{2m+1} P_{m-1}(x) + m P_{m-1}(x)$$

$$= \frac{-m(m+1)}{2m+1} P_{m+1}(x) + \frac{m(m+1)}{2m+1} P_{m-1}(x) = \frac{m(m+1)}{2m+1}\left[-P_{m+1}(x) + P_{m-1}(x)\right]$$

then

$$(1-x^2) P_m'(x) P_n'(x) = \frac{m(m+1)}{2m+1}\left[-P_{m+1}(x) P_n'(x) + P_{m-1}(x) P_n'(x)\right]$$

and

$$\int_{-1}^{1} (1-x^2) P_m'(x) P_n'(x) dx = \frac{m(m+1)}{2m+1}\left[-\int_{-1}^{1} P_{m+1}(x) P_n'(x) dx + \int_{-1}^{1} P_{m-1}(x) P_n'(x) dx\right]$$

$$= \frac{m(m+1)}{2m+1}\left[-P_{m+1}(x)P_n(x)\Big|_{-1}^{1} + \int_{-1}^{1} P'_{m+1}(x)P_n(x)dx \right.$$

$$\left. + P_{m-1}(x)P_n(x)\Big|_{-1}^{1} - \int_{-1}^{1} P'_{m-1}(x)P_n(x)dx \right]$$

$$= \frac{m(m+1)}{2m+1}\left[\int_{-1}^{1} P'_{m+1}(x)P_n(x)dx - \int_{-1}^{1} P'_{m-1}(x)P_n(x)dx \right]$$

$$= \frac{m(m+1)}{2m+1}\left[\int_{-1}^{1} P_n(x)\left[P'_{m+1}(x) - P'_{m-1}(x)\right]dx \right]$$

$$= \frac{m(m+1)}{2m+1}\left[\int_{-1}^{1} (2m+1)P_m(x)P_n(x)dx \right]$$

$$= \frac{m(m+1)}{2m+1}.(2m+1).\frac{2}{2m+1}\delta_{mn} = \frac{2m(m+1)}{2m+1}.\delta_{mn}$$

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(3) If $\boxed{u_n = \int_{-1}^{1} x^{-1}P_n(x)P_{n-1}(x)dx}$

show that $nu_n + (n-1)u_{n-1} = 2$ and hence evaluate u_n.

Answer

Since $xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x)$

Then

$$xP_{n-1}(x) = \frac{n}{2n-1}P_n(x) + \frac{n-1}{2n-1}P_{n-2}$$

$$xP_{n-1}(x)P_{n-1}(x) = \frac{n}{2n-1}P_n(x)P_{n-1}(x) + \frac{n-1}{2n-1}P_{n-2}P_{n-1}(x)$$

$$P_{n-1}(x)P_{n-1}(x) = \frac{n}{2n-1}x^{-1}P_n(x)P_{n-1}(x) + \frac{n-1}{2n-1}x^{-1}P_{n-2}P_{n-1}(x)$$

$$(2n-1)P_{n-1}(x)P_{n-1}(x) = nx^{-1}P_n(x)P_{n-1}(x) + (n-1)x^{-1}P_{n-2}P_{n-1}(x)$$

$$(2n-1)\int_{-1}^{1} P_{n-1}(x)P_{n-1}(x)dx = n\int_{-1}^{1} x^{-1}P_n(x)P_{n-1}(x)dx + (n-1)\int_{-1}^{1} x^{-1}P_{n-2}P_{n-1}(x)dx$$

$$\therefore (2n-1)\frac{2}{(2n-1)} = n\ u_n + (n-1)u_{n-1}$$

or $\quad n\ u_n + (n-1)u_{n-1} = 2$

at $n=1 \qquad u_1 = 2$

at $n=2 \qquad 2u_2 + u_1 = 2 \qquad \rightarrow u_2 = 0$

at $n=3 \qquad 3u_3 + 2u_2 = 2 \qquad \rightarrow u_3 = \frac{2}{3}$

$$u_n = \begin{cases} \frac{2}{n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

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(4) Show that

$$(a)\int_{-1}^{1} (x^2-1)P'_m(x)P_{m-1}(x)dx = \frac{-2m(m+1)}{4m^2-1}$$

$$(b)\int_{-1}^{1} xP_m(x)P_{m+1}(x)dx = \frac{2m(m+1)}{(2m+1)(2m+3)}$$

$$(c)\int_{-1}^{1} xP_m(x)P_{m-1}(x)dx = \frac{2m}{4m^2-1}$$

Answer:

$$(a)\int_{-1}^{1} (x^2-1)P'_m(x)P_{m-1}(x)dx = \frac{-2m(m+1)}{4m^2-1}$$

since $(x^2-1)P'_m(x) = mxP_m(x) - mP_{m-1}(x)$ then

$$\int_{-1}^{1} (x^2-1)P_m'(x)P_{m-1}(x)dx = \int_{-1}^{1} mxP_m(x) - mP_{m-1}(x)\ P_{m-1}(x)dx$$

$$= \int_{-1}^{1} mxP_m(x)P_{m-1}(x) - mP_{m-1}(x)P_{m-1}(x)\ dx$$

$$= m\int_{-1}^{1} xP_m(x)P_{m-1}(x)dx - m\int_{-1}^{1} P_{m-1}(x)P_{m-1}(x)dx \qquad (2)$$

since $xP_m(x) = \frac{m+1}{2m+1}P_{m+1}(x) + \frac{m}{2m+1}P_{m-1}(x)$

then (2) becomes

$$\int_{-1}^{1} (x^2-1)P_m'(x)P_{m-1}(x)dx$$

$$= m\int_{-1}^{1}\left[\frac{m+1}{2m+1}P_{m+1}(x) + \frac{l}{2m+1}P_{m-1}(x)\right]P_{m-1}(x)dx - \frac{2m}{2m-1}$$

$$= \frac{m(m+1)}{2m+1}\int_{-1}^{1} P_{m+1}(x)P_{m-1}(x)dx + \frac{m^2}{2m+1}\int_{-1}^{1} P_{m-1}(x)P_{m-1}(x)dx - \frac{2m}{2m-1}$$

$$= \frac{m^2}{2m+1}\int_{-1}^{1} P_{m-1}(x)P_{m-1}(x)dx - \frac{2m}{2m-1}$$

$$= \frac{m^2}{2m+1}\left[\frac{2}{2m-1}\right] - m\left[\frac{2}{2m-1}\right] = \frac{-2m(m+1)}{4m^2-1}$$

we use

$$\int_{-1}^{1} P_m(x)P_m(x)dx = \frac{2}{2m+1} \quad and \quad \int_{-1}^{1} P_{m-1}(x)P_{m-1}(x)dx = \frac{2}{2(m-1)+1}$$

<u>Another solution</u>

since $(x^2-1)P_m'(x) = (m+1)P_{m+1}(x) - x(m+1)P_m(x)$

then

$$\int_{-1}^{1} (x^2-1)P_m'(x)P_{m-1}(x)dx = \int_{-1}^{1}\left[(m+1)P_{m+1}(x) - x(m+1)P_m(x)\right]P_{m-1}(x)dx$$

$$= \int_{-1}^{1} \left[(m+1)P_{m+1}(x)P_{m-1}(x) - (m+1)xP_m(x)P_{m-1}(x)\right]dx$$

$$= (m+1)\int_{-1}^{1} P_{m+1}(x)P_{m-1}(x)dx - (m+1)\int_{-1}^{1} xP_m(x)P_{m-1}(x)dx$$

$$= -(m+1)\int_{-1}^{1} xP_m(x)P_{m-1}(x)dx \qquad (2)$$

since $xP_m(x) = \frac{m+1}{2m+1}P_{m+1}(x) + \frac{m}{2m+1}P_{m-1}(x)$ then (2) becomes

$$\int_{-1}^{1} (x^2-1)P'_m(x)P_{m-1}(x)dx$$

$$= -(m+1)\int_{-1}^{1} \left[\frac{m+1}{2m+1}P_{m+1}(x) + \frac{m}{2m+1}P_{m-1}(x)\right]P_{m-1}(x)dx$$

$$= -\frac{(m+1)(m+1)}{2m+1}\int_{-1}^{1} P_{m+1}(x)P_{m-1}(x)dx - \frac{m(m+1)}{2m+1}\int_{-1}^{1} P_{m-1}(x)P_{m-1}(x)dx$$

$$= -\frac{m(m+1)}{2m+1}\int_{-1}^{1} P_{m-1}(x)P_{m-1}(x)dx = -\frac{m(m+1)}{(2m+1)}\frac{2}{(2m-1)} = \frac{-2m(m+1)}{4m^2-1}$$

$$(b)\int_{-1}^{1} xP_m(x)P_{m+1}(x)dx = \frac{2m(m+1)}{(2m+1)(2m+3)}$$

Answer:

$$\int_{-1}^{1} xP_m(x)P_{m+1}(x)dx = \int_{-1}^{1} \left[\frac{m+1}{2m+1}P_{m+1}(x) + \frac{m}{2m+1}P_{m-1}(x)\right]P_{m+1}(x)dx$$

$$= \frac{m+1}{2m+1}\int_{-1}^{1} P_{m+1}(x)P_{m+1}(x)dx + \frac{m}{2m+1}\int_{-1}^{1} P_{m-1}(x)P_{m+1}(x)dx$$

$$= \frac{m+1}{2m+1}\int_{-1}^{1} P_{m+1}(x)P_{m+1}(x)dx + 0 = \frac{2m(m+1)}{(2m+1)(2m+3)}$$

$$(c)\int_{-1}^{1} xP_m(x)P_{m-1}(x)dx = \frac{2m}{4m^2-1}$$

Answer

$$\int_{-1}^{1} xP_m(x)P_{m-1}(x)dx = \int_{-1}^{1}\left[\frac{m+1}{2m+1}P_{m+1}(x)+\frac{m}{2m+1}P_{m-1}(x)\right]P_{m-1}(x)dx$$

$$= \frac{m+1}{2m+1}\int_{-1}^{1} P_{m+1}(x)P_{m-1}(x)dx + \frac{m}{2m+1}\int_{-1}^{1} P_{m-1}(x)P_{m-1}(x)dx$$

$$= 0 + \frac{m}{2m+1}\int_{-1}^{1} P_{m-1}(x)P_{m-1}(x)dx = \frac{2m}{(2m+1)(2m-1)} = \frac{2m}{(4m^2-1)}$$

---

(5) Show that $(1-x)\sum_{m=0}^{n}(2m+1)P_m(x) = (n+1)\{P_n(x)-P_{n+1}(x)\}$.

Answer:

since

$$(m+1)P_{m+1}(x)-(2m+1)xP_m(x)+mP_{m-1}(x)=0$$

then

$$(m+1)P_{m+1}(x)=(2m+1)xP_m(x)-mP_{m-1}(x) \quad (1)$$

$$(m+1)P_{m+1}(y)=(2m+1)yP_m(y)-mP_{m-1}(y) \quad (2)$$

Multiply (1) by $Pm(y)$ and (2) by $P_m(x)$

$$(m+1)P_{m+1}(x)P_m(y)=(2m+1)xP_m(x)P_m(y)-mP_{m-1}(x)P_m(y) \quad (3)$$

$$(m+1)P_{m+1}(y)P_m(x)=(2m+1)yP_m(y)P_m(x)-mP_{m-1}(y)P_m(x) \quad (4)$$

subtract (4) – (3)

$$(m+1)\left[P_{m+1}(y)P_m(x)-P_{m+1}(x)P_m(y)\right]$$

$$=(2m+1)(y-x)P_m(x)P_m(y)+m\left[P_m(y)P_{m-1}(x)-P_{m-1}(y)P_m(x)\right] \quad (5)$$

put $F_m = (m+1)\left[P_{m+1}(y)P_m(x)-P_{m+1}(x)P_m(y)\right]$

Then $F_0 = \left[P_1(y)P_0(x) - P(x)P_0(y)\right] = y - x$

and equation (5) can written in the form

$$F_m = (2m+1)\left(y-x\right)P_m(x)P_m(y) + F_{m-1} \qquad (6)$$

$$\sum_{m=0}^{n} F_m = \sum_{m=0}^{n} (2m+1)\left(y-x\right)P_m(x)P_m(y) + \sum_{m=0}^{n} F_{m-1} \qquad (7)$$

$$\sum_{m=0}^{n} F_m = \sum_{m=0}^{n} (2m+1)\left(y-x\right)P_m(x)P_m(y) + \sum_{m=1}^{n} F_{m-1} \qquad (8)$$

$$\sum_{m=0}^{n} F_m = \sum_{m=0}^{n} (2m+1)\left(y-x\right)P_m(x)P_m(y) + \sum_{m=0}^{n-1} F_m \qquad (9)$$

hence

$$\sum_{m=0}^{n} F_m - \sum_{m=0}^{n-1} F_m = \sum_{m=0}^{n} (2m+1)\left(y-x\right)P_m(x)P_m(y) \qquad (10)$$

$$F_n = \left(y-x\right)\sum_{m=0}^{n} (2m+1)P_m(x)P_m(y) \qquad (10)$$

Hence

$$(n+1)\left[P_{n+1}(y)P_n(x) - P_{n+1}(x)P_n(y)\right] = (y-x)\sum_{m=0}^{n} (2m+1)P_m(x)P_m(y)$$

put $y = 1$

$$(n+1)\left[P_{n+1}(x)P_n(1) - P_{n+1}(1)P_n(x)\right] = \left(1-x\right)\sum_{m=0}^{n} (2m+1)P_m(1)P_m(x)$$

$$(n+1)\left[P_n(x) - P_{n+1}(x)\right] = \left(1-x\right)\sum_{m=0}^{n} (2m+1)P_m(x)$$

---

(6) Show that $\boxed{\sum_{r=0}^{n} (2r+1)P_r(x) = P'_{n+1}(x) + P'_n(x)}$

Answer

$$P'_n(x) = \sum_{r=0}^{\frac{1}{2}[n-1]} (2n-4r-1)P_{n-2r-1}(x)$$

$$P'_{n+1}(x) = \sum_{r=0}^{\frac{1}{2}[n]} (2n-4r+1)P_{n-2r}(x)$$

$$P'_n(x) + P'_{n+1}(x) = \sum_{r=0}^{\frac{1}{2}[n-1]} (2n-4r-1)P_{n-2r-1}(x) + \sum_{r=0}^{\frac{1}{2}[n]} (2n-4r+1)P_{n-2r}(x)$$

$$= (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + \ldots + 7P_3(x) + 3P_1(x)$$

$$+ (2n+1)P_n(x) + (2n-3)P_{n-2}(x) + \ldots + 9P_4(x) + 5P_2(x) + P_0(x)$$

$$= \sum_{r=0}^{n} (2r+1)P_r(x)$$

---

(7) Show that $P_{2n}(0) = (-1)^n \dfrac{(2n)!}{2^{2n}(n!)^2}$.

Answer

Generating function $\dfrac{1}{\sqrt{(1-2xt+t^2)}} = \sum_{m=0}^{\infty} t^m P_m(0)$

set $x = 0$ in theorem (1) we obtain

$$\frac{1}{\sqrt{1+t^2}} = \sum_{m=0}^{\infty} t^m P_m(0)$$

Expanding the left-hand side by the binomial theorem gives us

$$\frac{1}{\sqrt{1+t^2}} = [1+t^2]^{-1/2} = 1 + (-1/2)t^2 + \frac{(-1/2)(-3/2)}{2!}(t^2)^2 + \ldots$$

$$+ \frac{(-1/2)(-3/2)\ldots(-\frac{2k-1}{2})}{k!}(t^2)^k + \ldots$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{1.2.3.4.5\ldots(2m-1)2m}{2^m m!\, 2.4.6\ldots.(2m-2)2m} t^{2m} = \sum_{m=0}^{\infty} (-1)^m \frac{(2m)!}{2^{2m}(m!)^2} t^{2m}$$

Thus we have

$$\sum_{m=0}^{\infty} (-1)^m \frac{(2m)!}{2^{2m}(m!)^2} t^{2m} = \sum_{m=0}^{\infty} t^m P_m(0)$$

and equating coefficients of corresponding powers of t on both sides gives $P_{2m}(0)=(-1)^m \dfrac{(2m)!}{2^{2m}(m!)^2}$ and $P_{2m+1}(0)=0$

---

(8) Show that $P_{2n}(0)-P_{2n-2}(0)=(-1)^n \dfrac{(2n)!}{2^{2n}(n)!^2}\left(\dfrac{4n-1}{2n-1}\right)$

Answer

Since $P_{2n}(0)=(-1)^n \dfrac{(2n)!}{2^{2n}(n!)^2}$ then $P_{2n-2}(0)=(-1)^{n-1}\dfrac{(2n-2)!}{2^{2n-2}(n-1)!^2}$

$$P_{2n}(0)-P_{2n-2}(0)=(-1)^n \frac{(2n)!}{2^{2n}(n!)^2}-(-1)^{n-1}\frac{(2n-2)!}{2^{2n-2}(n-1)!^2}$$

$$=(-1)^n \frac{(2n)!}{2^{2n}(n)!^2}+(-1)^n \frac{n^2(2n-1)(2n-2)!}{2^{2n-2}(2n-1)n^2(n-1)!^2}$$

$$=(-1)^n \frac{(2n)!}{2^{2n}(n)!^2}+(-1)^n \frac{2n.2n(2n-1)(2n-2)!}{2^{2n}(2n-1)n^2(n-1)!^2}$$

$$=(-1)^n \frac{(2n)!}{2^{2n}(n)!^2}+(-1)^n \frac{2n.(2n)!}{2^{2n}(2n-1)(n)!^2}$$

$$=\left((-1)^n \frac{(2n)!}{2^{2n}(n)!^2}\right)\left(1+\frac{2n}{2n-1}\right)=\left((-1)^n \frac{(2n)!}{2^{2n}(n)!^2}\right)\left(\frac{4n-1}{2n-1}\right)$$

---

(9) Use $P_0(x)=1, P_1(x)=x$ and the recurrence relation

$(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x)$ to find $P_2(x), P_3(x), P_4(x)$ and $P_5(x)$

Answer

Since $(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x)$

Setting n=1 then

$3xP_1(x)=2P_2(x)+P_0(x)$

$3x(x)=2P_2(x)+1 \qquad \Rightarrow \quad P_2(x)=\frac{1}{2}(3x^2-1)$

Setting n=2 then

$$5xP_2(x) = 3P_3(x) + 2P_1(x)$$

$$5x.\tfrac{1}{2}(3x^2 - 1) = 3P_3(x) + 2x$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

Setting n=3 then

$$7xP_3(x) = 4P_4(x) + 3P_2(x)$$

$$7x.\tfrac{1}{2}(5x^3 - 3) = 4P_4(x) + 2x$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

---

(10) Use Rodrigues formula to prove $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$.

Answer

From Rodrigues' formula we have

$$P_{n+1}(x) = \frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^{n+1} = \frac{1}{2^{n+1}(n+1)!}\frac{d^n}{dx^n}\left[2x(n+1)(x^2-1)^n\right]$$

$$= \frac{2(n+1)}{2^{n+1}(n+1)!}\frac{d^n}{dx^n}\left[x(x^2-1)^n\right] = \frac{1}{2^n(n)!}\frac{d^n}{dx^n}\left[x(x^2-1)^n\right]$$

$$P'_{n+1}(x) = \frac{1}{2^n(n)!}\frac{d^n}{dx^n}\left[(x^2-1)^n + 2x^2n(x^2-1)^{n-1}\right]$$

From Rodrigues' formula we have

$$P'_{n-1}(x) = \frac{d}{dx}\left[\frac{1}{2^{n-1}(n-1)!}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1}\right] = \frac{2n}{2^n n!}\frac{d^n}{dx^n}\left((x^2-1)^{n-1}\right)$$

As a consequence, we have.

$$P'_{n+1}(x) - P'_{n-1}(x) = \frac{1}{2^n(n)!}\frac{d^n}{dx^n}\left[(x^2-1)^n + 2x^2n(x^2-1)^{n-1}\right] - \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(2n(x^2-1)^{n-1}\right)$$

$$= \frac{1}{2^n(n)!}\frac{d^n}{dx^n}\left[(x^2-1)^n + 2x^2n(x^2-1)^{n-1} - 2n(x^2-1)^{n-1}\right]$$

$$=\frac{1}{2^n(n)!}\frac{d^n}{dx^n}\left[(x^2-1)^n+2n(x^2-1)(x^2-1)^{n-1}\right]$$

$$=\frac{2n+1}{2^n(n)!}\frac{d^n}{dx^n}(x^2-1)^n=(2n+1)P_n(x)$$

---

(11) Write x^2 as a linear combination of $P_0(x), P_1(x)$ and $P_2(x)$.

Answer

Let $x^2=C_0P_0+C_1P_1+C_2P_2=C_0+C_1x+C_2\frac{1}{2}(3x^2-1)$

Compare the coefficients in both sides then

$$x^2=C_0P_0+C_1P_1+C_2P_2$$

$$=C_0+C_1x+C_2\tfrac{1}{2}(3x^2-1)$$

$$C_0-C_2=0, C_1=0, \tfrac{3}{2}C_2=1$$

$$\therefore x^2=\tfrac{2}{3}P_0+0.P_1+\tfrac{2}{3}P_2$$

---

(12) Write x^3 as a linear combination of $P_0(x), P_1(x), P_2(x)$ and $P_3(x)$.

Answer

Let $x^3=C_0P_0+C_1P_1+C_2P_2+C_3P_3$

$$C_n=\frac{1}{2}(2n+1)\int_{-1}^{1}f(x)P_n(x)dx$$

For even integration $C_n=(2n+1)\int_0^1 f(x)P_n(x)dx$ then $C_0=C_2=0$

$$C_1=3\int_0^1 x^3P_1(x)dx=3\int_0^1 x^4dx=\tfrac{3}{5}$$

$$C_3=7\int_0^1 x^3P_3(x)dx=\frac{7}{2}\int_0^1 x^3\left(5x^3-3x\right)dx=\frac{7}{2}\int_0^1 x^3\left(5x^6-3x^4\right)dx=\tfrac{5}{2}$$

Then $x^3 = \frac{3}{5}P_1 + \frac{2}{5}P_3$

---

(13) Use (5),(6) and (7) to find $\int_{-1}^{1} x^2 P_2(x)dx$ and $\int_{-1}^{1} x^3 P_1(x)dx$.

The first several Legendre polynomials are listed below

$\mathbf{P_0(x) = 1}$

$\mathbf{P_1(x) = x}$

$\mathbf{P_2(x) = \frac{1}{2}(3x^2 - 1)}$

$\mathbf{P_3(x) = \frac{1}{2}(5x^3 - 3x)}$

$\mathbf{P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)}$

$\mathbf{P_5(x) = \frac{1}{8}(63x^5 - 70x3 + 15x)}$

The recurrence formula is

$$\mathbf{P_{n+1}(x) = \frac{2n+1}{n+1} xP_n(x) - \frac{n}{n+1} P_{n-1}(x)}$$

$$\mathbf{P'_{n+1}(x) - P'_{n-1}(x) = (2n+2)P_n(x)}$$

can be used to obtain higher order polynomials. In all cases $\mathbf{P_n(1) = 1}$ and $\mathbf{P_n(-1) = (-1)^n}$

Orthogonal Series of Legendre Polynomials

Any function $f(x)$ which is finite and single-valued in the interval $-1 \le x \le 1$, and which has a finite number or discontinuities within this interval can be expressed as a series of Legendre polynomials.

We let

$$f(x) = \sum_{n=0}^{\infty} A_n P_n(x) = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) + \ . \ . \ . \ , \quad -1 < x < 1$$

Multiplying both sides by $\mathbf{P_m(x)}$ and integrating with respect to x from $x = -1$ to $x = 1$

gives $\int_{-1}^{1} f(x) P_m(x)dx = \sum_{n=0}^{\infty} A_n \int_{-1}^{1} P_m(x) P_n(x)dx$

By means of the orthogonality property of the Legendre polynomials we can write

$$An = \frac{2n + 1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx\,, \qquad n = 0, 1, 2, 3 \ldots$$

Since $P_n(x)$ is an even function of x when n is even, and an odd function when n is odd, it follows that if $f(x)$ is an even function of x the coefficients A_n will vanish when n is odd; whereas if $f(x)$ is an odd function of x, the coefficients A_n will vanish when n is even. Thus for and even function $f(x)$ we have

$$A_n = \begin{cases} 0 & n \text{ is odd} \\ \dfrac{2n + 1}{2}\displaystyle\int_{-1}^{1} f(x)P_n(x)\,dx & n \text{ is even} \end{cases}$$

Whereas for an odd function $f(x)$ we have

$$A_n = \begin{cases} \dfrac{2n + 1}{2}\displaystyle\int_{-1}^{1} f(x)P_n(x)\,dx & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

---

(1) Find the first three coefficients in the expansion of the function

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ x & 0 < x < 1 \end{cases}$$

in a series of Legendre polynomials $P_n(x)$ over the interval $(-1, 1)$.

Solution

$$f(x) = \sum_{n=0}^{\infty} A_n P_n(x)$$

$$A_n = \frac{2n + 1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx\,, \qquad n = 0, 1, 2, 3$$

$$A_0 = \frac{2n + 1}{2}\int_{-1}^{1} f(x)P_0(x)\,dx = \frac{1}{2}\int_{0}^{1} (x)(1)\,dx = \frac{1}{4}$$

$$A_1 = \frac{2 + 1}{2}\int_{-1}^{1} f(x)P_1(x)\,dx = \frac{3}{2}\int_{0}^{1} (x)(x)\,dx = \frac{1}{2}$$

$$A_2 = \frac{4 + 1}{2}\int_{-1}^{1} f(x)P_2(x)\,dx = \frac{5}{2}\int_{0}^{1} x\left(\frac{1}{2}(3x^2 - 1)\right)dx = \frac{5}{4}\int_{0}^{1} (3x^3 - x)dx = \frac{5}{16}$$

$$f(x) = \frac{1}{4}P_0 + \frac{1}{2}p_1 + \frac{5}{16}P_2 + \ldots.$$